COGNITIVE TENDENCIES AND GENERATING MEANING IN THE ACQUISITION OF ALGEBRAIC SUBSTITUTION AND COMPARISON METHODS

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We studied the progress of algebraic syntax, once students have overcome the initial obstacles of the transition toward symbolic algebra. We analyzed the progress on the line of operation of the unknown, but when said unknown is represented by an expression involving a second unknown. One of the first times in the curriculum that this situation arises is when students learn the methods used to solve two-unknown linear equation systems. During the process of acquiring such methods, the cognitive tendencies identified in operation of a single unknown reappear (Filloy and Rojano 1989) and the need to re-elaborate the notion of algebraic equality becomes patent.

The transition processes involved in moving to algebraic thought have attracted the attention of many researchers dealing with the didactics of algebra. Studies such as those carried out by C. Kieran (1981), E. Filloy and T. Rojano (1989), R. Herscovics and L. Linchevsky (1991), A. Sfard and L. Linchevsky (1994), K. Stacey and M. MacGregor (1997), A. Gallardo (2002) and J. Vlasiss (2002), inter alia, have provided evidence to the effect that said transition involves profound changes in the mathematical thoughts of students. This research report broaches the topic of progressing in algebraic syntax, once students have overcome the first obstacles inherent in the transition toward symbolic algebra. We have specifically analyzed that progress on the line of the study Operating the Unknown (Filloy and Rojano, 1989), when the unknown is represented by an expression that involves a second unknown. One of the first times in a curriculum that this situation appears is when classic algebraic methods for solution of systems with two linear equations with two unknowns are introduced: the substitution and comparison methods.

In the two-unknown two-linear equation system: \( y = 2x + 3 \); \( y = 4x + 1 \), the student will have to operate the unknowns with “second level” representations. That is to say, in the example provided above in addition to being represented by a letter (the \( y \)) unknown \( y \) is also represented by an expression that involves the other unknown (the \( x \)). In the process of acquiring the new algebraic syntax, the cognitive tendencies identified in operation of a single unknown reappear (Filloy and Rojano, 1989; Filloy, 1991). In this research report, we will show that essential elements of algebraic representation must be reconstructed in order to acquire the sense of use of the methods of substitution and comparison (Filloy, Rojano and Solares, 2003; Filloy, Rojano & Puig, 2007, pp. 27-57).

THEORETICAL FRAMEWORK

The theoretical perspective adopted for this study was that of Local Theoretical Models (Kieran & Filloy, 1989; Filloy, Rojano, & Puig, 2007). According to said
perspective, we determined the essential components of teaching and learning methods for solving equation systems: the Teaching Model; the Cognitive Processes Model, though which learning processes are interpreted; the Formal Competence Model, which describes formal mathematical knowledge dealing with equation systems; and the Communication Model, by way of which message exchanges undertaken by the subjects are interpreted. This paper deals specifically with the components of the Formal Competence Model and the Cognitive Processes Model.

The Formal Model designed made it possible to define the transformations and meanings involved in applying the comparison and substitution methods. The list of transformations was defined based on the algebraic syntax work of D. Kirshner (1987), which deals with symbolic algebraic language. From Kirshner’s work, we incorporated the generation of simple algebraic expressions (additions, subtractions, multiplications, divisions and number and literal exponentiation) and the list of their transformations (the rules of associativity, commutativity, distributivity and multiplication and factorization of quadratic polynomials), which enable simplification of numerical operations and algebraic expressions. We added single-unknown linear equations and two-unknown linear equation systems to the expressions generated by Kirshner. We moreover added classic algebraic transformations that make it possible to operate the unknown in equations and systems: Transposition and Cancellation of terms, for single-unknown linear equations; and Substitution and Equalization of expressions for two-unknown two-linear equation systems. With respect to the theoretical elements of semantics, we took up the notions of Sinn (sense) and of Bedeutung (reference) as developed by G. Frege (1996) for the case of names. In an equation of the $y = Ax + B$ type, expression $Ax + B$ can be considered a “name” for the unknown. The reference for the latter expression results from its numerical value (unknown), while its sense is the mode in which that numerical value is expressed. In other words, it is the “chain of operations” that must be made in order to find the resulting value (For more information concerning this Formal Model, please see Rojano, 2005).

In the study Operating the Unknown (Filloy and Rojano, 1989), several cognitive tendencies were identified and characterized in student productions during their initial contact with operating unknowns. The set of these cognitive tendencies constitutes a model for the cognitive processes of this study. The following are several of the cognitive tendencies we identified: the return to more concrete situations when an analysis situation arises; focusing on readings made at language levels that will not enable solving the problem situation; and the presence of semantics-derived obstructions on syntax, and vice versa (Filloy & Rojano, 1989; Filloy, 1991; Filloy, Rojano, & Puig, 2007, pp. 163-189).

THE EXPERIMENTAL DESIGN

Clinical interviews were carried out with 12 secondary school students (aged 13 to 15). The students had been introduced to elementary algebra on the topic of solving single-unknown linear equations, but had not yet been taught how to solve equation systems. We chose seven students who systematically used the algebraic transformations of
Transposition or Cancellation of terms in order to solve single-unknown equations. We also selected another five students who had not yet consolidated their knowledge of algebraic syntax for operation of single-unknown equations, but who did have a high level of competence in numerical calculation and systematically used arithmetic strategies, such as Trial and Error, in order to solve single-unknown equations.

The interview script was designed based on an analysis undertaken within the Formal Model of the comparison and substitution methods. The different transformations applied to solving a two-unknown two-linear equation system are introduced in the script in order to take student knowledge to its very “limits”. That is to say, once understanding of the problem of finding a solution to an equation system is guaranteed, their knowledge is taken to a point at which that knowledge is no longer sufficient enough for the students to solve the systems presented. Variation of the numerical domains of the solutions (natural numbers, decimals, fractions, positives and negatives) and of the syntactic structures of the equations generate “obstructions” in the interpretation and operation of the different expressions of the unknown. Equalization and Substitution were introduced as the means of operating the unknown in this new level of representation. Table 1 shows the list of items presented. Solutions are presented as \((x, y)\).

<table>
<thead>
<tr>
<th>Equation 1</th>
<th>Equation 2</th>
<th>Solution 1</th>
<th>Solution 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = 4x + 2; 2x + 6 = y)</td>
<td>(4y - 1 = x; 2y + 7 = x)</td>
<td>((2, 10))</td>
<td>((15, 4))</td>
</tr>
<tr>
<td>(x = 5y + 1; x = 3y - 1)</td>
<td>(x = y + 1; x + y = 11)</td>
<td>((3, 6))</td>
<td>((4, 9))</td>
</tr>
<tr>
<td>(4x - 3 = y; 6x = y - 7)</td>
<td>(2x + 3y = 72; 3y = x)</td>
<td>((-5, -23))</td>
<td>((24, 8))</td>
</tr>
</tbody>
</table>

Table 1. List of items presented

RESULTS OF INTERVIEW ANALYSIS

Returning to more concrete levels and focusing on positive integers

In general terms and without significant differences between application of either the substitution or the comparison method, the students made use of more concrete levels than algebraic levels in order to operate the unknowns while solving the new problems. Their preferred spontaneous strategy was Trial and Error -applied by nine of the 12 students interviewed.

As the interview progressed, the variation of solution numerical domain generated great difficulties. The cognitive tendency to place themselves at more concrete levels became more acute. In spite of their high numerical competences (including operations with negative numbers), four students assumed that there would only be positive solutions and reached the conclusion that in systems of the type \([y = 2x;\)
$5x + 3 = y$, the numerical values of the expression $5x + 3$ would always be bigger than the numerical values of the expression $2x$. For instance, one of the students interviewed, Circe (C), sought the values of the unknowns in the following manner: (The letter I indicates the Interviewer).

C: It’s just that it doesn’t work, because if the $y$ is worth two $x$, it cannot be worth five $x$ plus three.

I: Why not?

C: If it’s worth two $x$, five $x$ plus three will always be bigger than two $x$.

I: Always?

C: Yes.

I: Let’s see. Why do you say that this will be greater than this?

C: Well, because five $x$ will always be bigger than two $x$.

I: Uh-huh. What about negative numbers?

C: With negative numbers too. $y = 2(-1)$

$y = -2$

C: It does work with negative numbers. Yes, five minus one, plus three is equal to minus two… [Writes]

$5(-1) + 3 = -2$

$y = 2(-1)$

$y = -2$

$-5 + 3 = -2$

$-2 = -2$

C: $y$ is worth… [Writes]

$y = -2$

$x = -1$

In this case, focusing on the domain of positive numbers for looking for the solutions of the equations and making the operations (additions, subtractions, multiplications and divisions) prevented Circe from finding the solution. At that point it became necessary to broaden the numerical domain by incorporating negative numbers. This need to broaden the numerical domain became necessary for the majority of the students interviewed.

The presence of syntax-derived obstructions

Five students demonstrated a non-symmetrical interpretation of equality. During the process of solving a system of the type $y = Ax + B; y = Cx + D$, for example, using the comparison method, the expressions $Ax + B$ and $Cx + D$ are equalized, and reduced equation $Ax + B = Cx + D$ is obtained. The non-symmetry mistake consists
of changing the sign in expression $Ax + B$, since it goes from being located on the right-hand side of the equal sign [in $y = Ax + B$] to the left-hand side [in $Ax + B = Cx + D$]. Let us take a look at non-symmetry in the case of Ana (A).

I: I have a question. One of your classmates was saying “this $x$, $x$ [points to $x$ in $x = 5y$] is equal to five $y$, and that $x$ [points to $x$ in $x = 54 - y$] is equal to fifty-four minus $y$”, so, like you, he said “this [points to $5y$] must be equal to this [points to $54 - y$]”, right? And he wrote down the equality, but, but your classmate said “be careful with the signs! This five $y$ [points to $5y$ in $x = 5y$] is on the right-hand side and here [points to the $x$ in $x = 54 - y$] we’re going to put it on the left-hand side”. Like you, right? [I points to the equation obtained from the equalization: $5y = 54 - y$]. “We’re going to put it on the left-hand side and since it’s going to change sides, the sign has to change…”

A: [Interrupts] To negative!

I: What do you think?

A: Yes.

I: Yes? Well, how would you do it?

A: [Writes]

\[-5y = 54 - \]

A: Fifty-four minus $y$?

I: What?

A: Yes, don’t you think?… Yes [completes the equation].

\[-5y = 54 - y \]

I: So that’s how you would do it then?

A: I just change the sign.

I: What sign?

A: The one for the five.

The error arises when Transposition of terms is applied in an over generalized manner to a case in which it is not an unknown that is being operated, rather it is a matter of an Equalization of expressions. Transposition is a transformation undertaken in order to operate an unknown within an equation, whereas the new transformation -the Equalization- is a transformation of equation systems that results in a single-unknown equation. The error corresponds to the cognitive tendency of obstructions derived from previously learned syntax. It was also found in the solutions obtained using the substitution method.

The presence of semantics-derived obstructions and the need to broaden the notion of algebraic equality

In three of the 12 cases, readings that were focused on the domain of positive solutions led to the manifestation of another cognitive tendency related to the analysis of the syntactic structure of the expressions (the superficial structures, following the work of D. Kirshner, 1987). While solving the system \[y = 2x; 5x + 3 = y\] by applying the comparison method, the students stated that the expressions could indeed be equal “because they are $x$ signs, it’s like if it was a five, two times $x$ [2x]
and… five times x \([5x + 3]\)”, but “with the plus sign it doesn’t work anymore… because plus three is yet another operation [referring to “+ 3” in \(5x + 3\)]… it gives another number”. For instance Raúl \(\Box(R\Box)\) stated the following:

R: I think you can’t do this one.
I: You can’t do it? Well, explain why not to me again.
R: It’s that it’s adding times three, I mean it’s adding three in addition to the five times x, that’s bigger than the two.
I: You told me that you had tried with… big numbers, with decimals too, with fractions. Have you tried it with negative numbers? You did one with minus three, right? How about with minus two?
R: No, this would be too big [points to the equation \(5x + 3 = y\)] and this one would be too small [points to the equation \(y = 2x\)].
I: And we were saying: This y is equal to this y [points to the ys in the two equations]. This x is equal to this x [points to the xs in the two equations], right? It says “y is equal to two x”, right? [Points out in the equation \(y = 2x\)]. And the y is equal to this [points to the equation \(5x + 3 = y\)]. What I was asking you is: Do you think it’s true that this is equal to this? [Points to \(5x + 3\) in the equation \(5x + 3 = y\) and points to \(2x\) in the equation \(y = 2x\)].
R: Not any more, not with the plus sign.
I: Not any more? Why not?
R: Because plus three is another operation altogether.
R: It gives another number.

In Raúl’s case, focusing on positive numbers generated conflicts surrounding the notion of equality: two expressions with different syntactic structures could not be equalized. In this conflict, the tendency of presence of numerical semantics-derived obstructions is manifested.

Generally speaking, manifestation of the cognitive tendencies described attests to the conflicts faced by students at this time of transition, moving from representation of one unknown to representation of an unknown given in terms of another unknown. Students have to operate different types of equalities and unknown representations. Let us now see how Carlos (Ca) does in his attempts to solve the system \([x = 5y, x = 54 - y]\).

I: I have a question for you, Carlos. Well… one of your classmates the other day was saying the following: “this x [points to x in the first equation of the system underway \(x = 5y\)] is equal to this x” [points to x in the second equation of the system: \(x = 54 - y\)]. Do you agree with that?
Ca: Uh-huh. [Nods].
I: And he was saying “this y [points to y in \(x = 5y\)] is equal to this y” [points to y in \(x = 54 - y\)].
Ca: Uh-huh… [He’s not convinced].
I: Right?
Ca: Here it’s five times [points to \(5y\) in \(x = 5y\)].
I: Here [points to \(5y\) in \(x = 5y\)] it’s five times …
Ca: But here, it’s just once [points to y in \(x = 54 - y\)].
I: Here, just once [points to y in x = 54 - y]… And he was saying “since this x is equal to five y [points to x in x = 5y] and this x is equal to fifty-four minus y [points to x in x = 54 - y] and the two xs are the same, then… I have”. He said “that five y is equal to fifty-four minus y” [points to 5y in x = 5y, and points to 54 - y in x = 54 - y]. Do you think that’s right?

Ca: No!

I: No?

Ca: No, well no!

I: Why not?

Ca: Because this is completely different [points to x = 5y] from this [points to “- y” in x = 54 - y] and from this [points to 54 in the equation x = 54 - y].

I: How are they different?

Ca: In everything, in their values and in everything.

I: Well… Does it matter that this is equal to this [points to x in the equation x = 5y] and that it says that the same thing [points to x in the equation x = 54 - y] is equal to this? [points to 54 - y in the equation x = 54 - y]. It says “x is equal to five y [points to the equation x = 5y]” and “x is equal to fifty-four minus y [points to the equation x = 54 - y]”. Are they the same or not?

Ca: No, well no they’re not.

I: Right? Then was your classmate right or not?

Ca: Yes.

I: Yes?

Ca: In this [points to x in the two equations: x = 5y, x = 54 - y], but not in this [points to “5y” and “54 - y”].

Two separate interpretations of algebraic equality are present in Carlos’ interpretation. In the system \([x = 5y, x = 54 - y]\) the value of \(x\) is equal to \(5y\) and, at the same time, it is equal to \(54 - y\). On the one hand, the identity of the representation is present: \(x\) is representing the same unknown value in both equations. Yet on the other hand, the restricted equality established between an unknown and the algebraic expression that corresponds to it through the equation is also present. The expressions \(5y\) and \(54 - y\) have the same numerical values, they have equal references. But at the same time, the values of those algebraic expressions are referred to in different ways: “Five y [5y] is not the same as fifty-four minus y [54 - y]”. The expressions have different senses.

**FINAL REMARKS**

The results obtained in this study confirm the presence of the cognitive tendencies found in the study *Operation of the Unknown* (Filloy and Rojano, 1989) in this new level of algebraic representation of the unknown.

In order to acquire competent use of the methods, that is to say, to acquire the sense of use of the methods (Filloy, Rojano, & Solares, 2003; Filloy, Rojano, & Puig, 2007, pp. 191-213), essential elements of algebraic representation must necessarily be reconstructed: the reference and the sense of the different algebraic expressions used to represent unknowns at this new level.
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References


