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Characterization of n-Path Graphs and of Graphs Having nth Root

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We characterize connected graphs and digraphs having an nth root and so generalize results by A. Mukhopadhyay and D. P. Geller, respectively. We then define the n-path graph of a graph and characterize those graphs which are n-path graphs. This extends recent results by B. Devadas Acharya and M. N. Vartak. The corresponding problem for digraphs is also considered.

1. GRAPHS AND DIGRAPHS WITH AN nTH ROOT

We consider only finite graphs and digraphs, without loops or multiple edges. $V(G)$ and $E(G)$ denote the vertex and edge set of the graph G. The edge with end vertices u and v will be denoted by uv. If no confusion arises, we write sometimes $u \in G$ and $uv \in G$ meaning respectively $u \in V(G)$ and $uv \in E(G)$. For $u, v \in G$ let $d_G(u, v)$ denote the distance between u and v. If $n \geqslant 1$, then the *nth power* of G, written G^n, is defined to be the graph having $V(G)$ as vertex set and two vertices u, v being adjacent in G^n if, and only if, $d_G(u, v) \leqslant n$. We say that the graph H has an *nth root* if there exists a graph G such that $G^n = H$. Corresponding definitions and notation for digraphs will be used. Undefined terms can be found in [5].

Mukhopadhyay [6] found the following characterization of graphs having square root.

THEOREM 1 (Mukhopadhyay). *A connected graph G with p vertices labeled $v_1, \ldots, v_p$ has a square root if, and only if, G contains a collection of p complete subgraphs $G_1, \ldots, G_p$ such that for all $i, j = 1, \ldots, p$*

(i) $v_i \in G_i$,

(ii) $v_i \in G_j \Leftrightarrow v_j \in G_i$,

(iii) $v_iv_j \in G \Rightarrow$ *there is a G_k containing v_iv_j.*

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In order to generalize this result for all $n \geqslant 2$, let the vertices of G be labeled $v_1, \dots, v_p$ and let Γ be a collection of p subgraphs $G_1, \dots, G_p$ of G. For two different vertices u, v of G, define a *u, v-linking L with respect to Γ* (or, shortly, a *u, v-linking*, if Γ is fixed) as a collection $\{G_{i_1}, \dots, G_{i_k}\}$ of $k \geqslant 1$ elements of Γ such that $v = v_{i_k}$, and $u = v_{i_0} \in G_{i_1}, v_{i_1} \in G_{i_2}, \dots, v_{i_{k-1}} \in G_{i_k}$. The number k of members of the collection is called the *length* of L. The *union UL* of a linking L of length k is the subgraph of G spanned by the union of the vertex sets of the first $k - 1$ members of the linking.

Now we have the following theorem.

THEOREM 2. *A connected graph G with p vertices labeled $v_1, \dots, v_p$ has an nth root, $n \geqslant 2$, if, and only if, G contains a collection $\Gamma = \{G_1, \dots, G_p\}$ of p complete subgraphs such that*

(i) $v_i \in G_i$ *for all* $i = 1, \dots, p$;

(ii) $v_i \in G_j \Leftrightarrow v_j \in G_i$ *for all* $i, j = 1, \dots, p$;

(iii) $uv \in G \Rightarrow$ *there exists a u, v-linking L (with respect to Γ) of length not exceeding n such that $uv \in UL$;*

(iv) *for all u, v-linkings L with length $\leqslant n$, the subgraph UL is complete.*

Proof. I. Necessity. Let $H^n = G$. Then define, for all $i = 1, \dots, p$, the set $\bar{G}_i = \{v_i\} \cup \{v_j \mid v_i v_j \in H\}$, and let G_i be the subgraph of G spanned by $\bar{G}_i$. We claim that $\Gamma = \{G_1, \dots, G_p\}$ satisfies conditions (i)–(iv).

Clearly, the G_i are complete, and it is obvious that (i) holds. Further, $v_i \in G_j \Leftrightarrow v_i \in \bar{G}_j \Leftrightarrow v_i v_j \in H \Leftrightarrow v_j \in \bar{G}_i \Leftrightarrow v_j \in G_i$, so (ii) also holds.

In order to prove (iii), let uv be an arbitrary edge of G. Then, since $G = H^n$, we have $d_H(u, v) \leqslant n$, and there exists a path

$$u = v_{i_0}, v_{i_1}, \dots, v_{i_k} = v$$

in H such that $k \leqslant n$. Thus $G_{i_1}, \dots, G_{i_k}$, because of the definition of the G_i, form a u, v-linking of length $k \leqslant n$ and whose union contains uv.

Finally, let $L = \{G_{i_1}, \dots, G_{i_k}\}$ be a v_{i_0}, v_{i_k}-linking with $k \leqslant n$. Evidently, the distance between any two points $u, v \in UL = \bigcup_{\nu=1}^{k-1} G_{i_\nu}$ does not exceed n in H, and therefore the points u, v are joined in $G = H^n$. Hence UL is complete and (iv) is proved.

II. Sufficiency. Let $\Gamma = \{G_1, \dots, G_p\}$ be a collection of p complete subgraphs of G satisfying the required conditions (i)–(iv). Define then the graph H having $V(G)$ as vertex set and v_i, v_j being adjacent if, and only if, $v_i \in G_j$ (and hence $v_j \in G_i$). We claim that $H^n = G$ and that the mapping $f(v_i) = v_i$ provides an isomorphism.

(a) Let $uv \in H^n$. Then $d_H(u, v) \leqslant n$ and there exists a path $u = v_{i_0}$, $v_{i_1}, \ldots, v_{i_k} = v$, $k \leqslant n$, in H. Then $v_{i_\nu} v_{i_{\nu+1}} \in H$ for all $\nu = 0, 1, \ldots, k-1$, and, therefore, according to the definition of H, $v_{i_0} \in G_{i_1}$, $v_{i_1} \in G_{i_2}, \ldots$, $v_{i_{k-1}} \in G_{i_k}$. This implies that $G_{i_1}, \ldots, G_{i_k}$ form a u, v-linking L of length $k \leqslant n$. Because of (iv), UL is complete and, since $u = v_{i_0} \in G_{i_1}$ and $v = v_{i_k} \in G_{i_{k-1}}$, we conclude that $uv \in G$.

(b) Take now $uv \in G$. Consider a v_{i_0}, v_{i_k}-linking $L = \{G_{i_1}, \ldots, G_{i_k}\}$ of length $k \leqslant n$ such that UL contains uv, as guaranteed by (iii). Without loss of generality, let $u \in G_{i_r}$ and $v \in G_{i_s}$ with $r \leqslant s$. According to the definition of H, $v_{i_r} v_{i_{r+1}}, v_{i_{r+1}} v_{i_{r+2}}, \ldots, v_{i_{s-1}} v_{i_s}$ are all edges of H, and, since $r \geqslant 1$ and $s \leqslant n-1$, we have that $d_H(v_{i_r}, v_{i_s}) \leqslant s - r \leqslant n-2$. If $u \neq v_{i_r}$ or $v \neq v_{i_s}$, then uv_{i_r} or vv_{i_s} are, respectively, edges of H, and therefore $d_H(u, v) \leqslant n$. But this means $uv \in H^n$.

The proof is now complete.

An analog of Theorem 2 for digraphs can easily be proved with the following notation introduced by Geller [4]. For a vertex v and two point sets S and T (not both empty) of a digraph D such that $v \notin S \cup T$, define a *carrier-complete digraph* $K(S, v, T)$ having vertex set $V = S \cup \{v\} \cup T$ and edge set $E = (S \times \{v\}) \cup (\{v\} \times T) \cup (S \times T)$. In [4] Geller proved the following analog for Theorem 1.

THEOREM 3 (Geller). *A connected digraph D with p vertices labeled $v_1, \ldots, v_p$ has a square root if, and only if, there exists a collection of p carrier-complete subdigraphs $K_i = K(S_i, v_i, T_i)$, where the K_i are associated in a one-to-one manner with the vertices v_i of D, such that*

(i) $v_i \in T_j \Leftrightarrow v_j \in S_i$;

(ii) *the K_i cover D.*

Given now a family $\Omega = \{K_1, \ldots, K_p\}$ of carrier-complete subdigraphs of a digraph D with vertices labeled $v_1, \ldots, v_p$ define a *u, v-carrier linking L* (*with respect to Ω*) as a collection $\{K_{i_1}, \ldots, K_{i_k}\}$ of $k \geqslant 1$ elements of Ω such that $v = v_{i_k}$ and $u = v_{i_0} \in S_{i_1}$, $v_{i_1} \in S_{i_2}, \ldots, v_{i_{k-1}} \in S_{i_k}$. As before, k is called the *length* of the carrier linking and the digraph spanned by the set $\bigcup_{\nu=1}^{k-1} K_{i_\nu}$ is called the *union* of the linking L and is denoted by UL. Then we have the following theorem.

THEOREM 4. *A connected digraph D with p labeled vertices $v_1, \ldots, v_p$ has an nth root ($n \geqslant 2$) if, and only if, D contains a collection $\Omega = \{K_1, \ldots, K_p\}$ of p carrier-complete subdigraphs $K_i = K(S_i, v_i, T_i)$ such that*

(i) $v_i \in T_j \Leftrightarrow v_j \in S_i$;

(ii) *uv is an edge of D if, and only if, there is a u, v-carrier linking of length* $\leqslant n$.

Proof. I. Necessity. Let $F^n = D$. Then define, for all $i = 1,..., p$, the sets $S_i = \{v_j \mid v_j v_i \in F\}$, $T_i = \{v_k \mid v_i v_k \in F\}$. By construction, the sets S_i and T_i, together with the vertex v_i span (in D) a carrier-complete subdigraph $K_i = K(S_i, v_i, T_i)$. We claim that $\Omega = \{K_1,..., K_p\}$ satisfies conditions (i)–(iii).

This is indeed immediate for (i) since $v_i \in T_j \Leftrightarrow v_j v_i \in F \Leftrightarrow v_j \in S_i$. If now uv is an arc of D, then $d_F(u, v) \leqslant n$ and there is a directed path $u = v_{i_0}, v_{i_1},..., v_{i_k} = v$ in F such that $k \leqslant n$. Thus $u \in S_{i_1}, v_{i_1} \in S_{i_2},...,$ $v_{i_{k-1}} \in S_{i_k}$, and $K_{i_1},..., K_{i_k}$ form a u, v-carrier linking of length $k \leqslant n$ containing uv, so (ii) holds.

Finally, let $L = \{K_{i_1},..., K_{i_k}\}$ be a v_{i_0}, v_{i_k}-carrier linking with $k \leqslant n$. Observe that if $u \in S_{i_r}$, $v \in T_{i_s}$ and $r \leqslant s$, for two points $u, v \in UL$, then u and v are joined by a directed path in F having length at most n, that is, $d_F(u, v) \leqslant n$, and therefore $uv \in D$. In particular $v_{i_0} v_{i_k} \in D$.

II. Sufficiency. Let $\Omega = \{K_1,..., K_p\}$ be a collection of p carrier-complete subdigraphs of D satisfying (i)–(iii). Define the digraph F on the vertex set of D having $v_i v_j$ as an arc if, and only if, $v_i \in S_j$ (and hence $v_j \in T_i$). We claim that $F^n = D$, and that the mapping $f(v_i) = v_i$ provides an isomorphism. But indeed, the following seven statements are equivalent (the equivalence of (f) and (g) following from (ii)):

(a) $uv \in F^n$;

(b) $d_F(u, v) \leqslant n$;

(c) there is a directed path $u = v_{i_0},..., v_{i_k} = v$ in F, $k \leqslant n$;

(d) $uv_{i_1}, v_{i_1} v_{i_2},..., v_{i_{k-1}} v$ are all arcs of F, all vertices are different and $k \leqslant n$;

(e) $u \in S_{i_1}, v_{i_1} \in S_{i_2},..., v_{i_{k-1}} \in S_{i_k}, v_{i_k} = v$, all vertices are different and $k \leqslant n$;

(f) $K_{i_1},..., K_{i_k}$ form a u, v-carrier linking in D, $u \in S_{i_1}$, $v = v_{i_k}$, $k \leqslant n$;

(g) $uv \in D$.

Now the proof is complete.

2. n-PATH GRAPHS

Let G be a finite connected graph. G_n be the graph having $V(G)$ as vertex set and for which two vertices u, v are adjacent if, and only if, there exists a u, v-path of length exactly n in G. We call G_n the *n-path graph* of G. The *n-path digraph* of a digraph is defined analogously.

In a recent paper [3] B. Devadas Acharya and M. N. Vartak define the *open neighborhood* O_u of a vertex $u \in G$ as the set of all vertices $v \in G$ adjacent to u. The *open neighborhood graph* $N(G)$ of G is then defined as the intersection graph of the family $\{O_u\}_{u \in G}$. Obviously, $N(G)$ is isomorphic to the two-path graph G_2 of G.

The following characterization of open neighborhood graphs, and hence of two-path graphs, is found by the authors in [3].

THEOREM 5 (B. Devadas Acharya and M. N. Vartak). *A connected graph G with vertices labeled* $v_1, ..., v_p$ *is open neighborhood graph of some graph H if, and only if, G contains a collection of complete subgraphs* $G_1, ..., G_p$ *such that for all* $i, j = 1, ..., p$

(i) $v_i \notin G_i$,

(ii) $v_i \in G_j \Leftrightarrow v_j \in G_i$,

(iii) $v_i v_j \in G \Rightarrow$ *there exists a* G_k *containing* $v_i v_j$.

The striking similarity of this characterization with the one for square roots presented in Theorem 1 led us to try a corresponding characterization for n-path graphs, $n \geqslant 2$. Indeed we proved the following theorem.

THEOREM 6. *A connected graph G with p vertices labeled* $v_1, ..., v_p$ *is the n-path graph* ($n \geqslant 2$) *of some graph H if, and only if, G contains a collection* $\Gamma = \{G_1, ..., G_p\}$ *of p subgraphs (not necessarily complete) such that*

(i) $v_i \notin G_i$ *for all* $i = 1, ..., p$;

(ii) $v_j \in G_i \Leftrightarrow v_i \in G_j$ *for all* $i, j = 1, ..., p$;

(iii) $uv \in G \Rightarrow$ *there exists a u, v-linking (with respect to* Γ*) of length n*;

(iv) *for any* v_{i_0}, v_{i_n}*-linking* $L = \{G_{i_1}, ..., G_{i_n}\}$ *of length n, uv is an edge of G whenever* $u \in G_{i_1} - v_{i_2}$ *and* $v \in G_{i_{n-1}} - v_{i_{n-2}}$.

Proof. I. Necessity. Let $G = H_n$. Define for every vertex v_i the set $\bar{G}_i = \{u \mid uv_i \in H\}$, let G_i be the subgraph of G spanned by $\bar{G}_i$ and let $\Gamma = \{G_i\}$.

Then, obviously, (i) holds. Further, $v_i \in G_j \Leftrightarrow v_i \in \bar{G}_j \Leftrightarrow v_i v_j \in H \Leftrightarrow v_j \in \bar{G}_i \Leftrightarrow v_j \in G_i$, so (ii) is proved.

Let now uv be an edge of G. Then there is an n-path $u = v_{i_0}, v_{i_1}, ..., v_{i_n} = v$ in H, and, according to the definition of the G_i, there is a u, v-linking (with respect to Γ) of length n, so (iii) holds.

Conversely, if $L = \{G_{i_1}, ..., G_{i_n}\}$ is a v_{i_0}, v_{i_n}-linking of length n, then $v_{i_0} \in G_{i_1}, v_{i_1} \in G_{i_2}, ..., v_{i_{n-1}} \in G_{i_n}$ and there exists an n-path $v_{i_0}, v_{i_1}, ..., v_{i_n}$

joining v_{i_0} and v_{i_n} in H. If now $u \in G_{i_1} - v_{i_2}$ (i.e., $uv_{i_1} \in H$) and $v \in G_{i_{n-1}} - v_{i_{n-2}}$ (i.e., $v_{i_{n-1}}v \in H$), then there is also an n-path $u, v_{i_1},..., v_{i_{n-1}}, v$ in H, and therefore uv is an edge of G. This proves (iv).

II. Sufficiency. Let G contain a family $\Gamma = \{G_1,..., G_p\}$ of subgraphs satisfying (i)–(iv). Construct the graph H with $V(H) = V(G)$ and $v_iv_j \in H$ if and only if $v_i \in G_j$ (and hence $v_j \in G_i$). We claim that $H_n = G$ and that the labeling of the vertices provides an isomorphism.

Indeed, take first $uv \in H_n$ and let $u = v_{i_0}, v_{i_1},..., v_{i_n} = v$ be an n-path in H joining u and v. But this means, according to the definition of H, that $u \in G_{i_1},..., v_{i_{n-1}} \in G_{i_n}$, i.e., that $G_{i_1},..., G_{i_n}$ constitute a u, v-linking of length n. Since being vertices of a path, $u \neq v_{i_2}$ and $v \neq v_{i_{n-2}}$, then it follows from (iv) that $uv \in G$. Thus $E(H_n) \subseteq E(G)$.

On the other hand, if uv is an edge of G, then (iii) ensures us the existence of a u, v-linking $\{G_{i_1},..., G_{i_n}\}$ in G. But then $u \in G_{i_1},..., v_{i_{n-1}} \in G_{i_n}$ and therefore $uv_{i_1}, v_{i_1}v_{i_2},..., v_{i_{n-1}}v$ are edges of H providing a u, v-path of length n in H, and in consequence implying $uv \in H_n$.

Thus $H_n = G$, and Theorem 6 is proved.

Using the same techniques, analogs of Theorems 5 and 6 can be derived. We present only the case $n = 2$.

Let S and T be two vertex sets of a digraph D, not necessarily disjoint; define a *one-way bipartite digraph* $B(S, T)$ as the subdigraph having vertex set $S \cup T$ and edge set $\{uv \mid u \in S, v \in T\}$. Then we have the following theorem.

THEOREM 7. *A connected digraph D with labeled vertices $v_1,..., v_p$ is the two-path digraph of some digraph F if, and only if, D contains a collection $\Delta = \{B_1,..., B_p\}$ of p one-way bipartite subdigraphs $B_i = B(S_i, T_i)$ satisfying for all $i, j = 1,..., p$*

(i) $v_i \notin B_i$;

(ii) $v_i \in B_j \Leftrightarrow v_j \in B_i$;

(iii) $v_iv_j \in D \Rightarrow$ *there is a* B_k *containing* v_iv_j.

Proof. Let first $F_2 = D$. Define for each $v_i \in F$ the sets $S_i = \{v_j \mid v_jv_i \in F\}$, $T_i = \{v_k \mid v_iv_k \in F\}$ and let B_i be the subdigraph of $D\,(=F_2)$ spanned by $S_i \cup T_i$. Clearly, $B_i = B(S_i, T_i)$ is a one-way bipartite subdigraph of D. Moreover, the B_i satisfy conditions (i)–(iii). For indeed, $v_i \notin B_i$ for all $i = 1,..., p$; $v_i \in S_j \Leftrightarrow v_iv_j \in F \Leftrightarrow v_j \in T_i$; and, if $v_iv_j \in D = F_2$, then there is a two-path v_i, v_k, v_j in F and, therefore, $v_i \in S_k$, $v_j \in T_k$, that is, $v_iv_j \in B_k$.

Conversely, let $\Delta = \{B_1,..., B_p\}$ be a set of one-way bipartite subdigraphs of D as required by conditions (i)–(iii). Define the digraph F on

the vertex set of D drawing an arc from v_i to v_j if, and only if, $v_i \in S_j$. We claim that $F_2 = D$ and that the labeling provides an isomorphism. Indeed, $v_i v_j \in D \Leftrightarrow$ there is a B_k containing $v_i v_j \Leftrightarrow v_i \in S_k$ and $v_j \in T_k \Leftrightarrow v_i \in S_k$ and $v_k \in S_j \Leftrightarrow v_i v_k$, $v_k v_j \in F \Leftrightarrow$ there is in F a 2-path from v_i to $v_j \Leftrightarrow v_i v_j \in F_2$.
This proves the theorem.

Note that in Theorems 1–7 the word "connected" could be supressed since the proofs can always be accomplished separately in the same way for each component. Nevertheless, we have prefered to carry it along in order to be consequent with the original versions of Theorems 1, 3 and 5.

3. Further Remarks

Following an idea of B. Devadas Acharya and M. N. Vartak [3], we want to prove the next theorem.

THEOREM 8. *For a given $n \geqslant 2$ any digraph D can be embedded (as induced subdigraph) in a digraph D' which is the n-path digraph of some digraph.*

Proof. Construct the digraph F as follows. Take the vertices of D. For every arc uv of D introduce a u, v-directed path of length n in F (i.e., for each uv introduce $n - 1$ new vertices and n arcs). Now, $uv \in D \Leftrightarrow$ there exists an n-path from u *to* v in $F \Leftrightarrow uv \in F_n$ and $u, v \in D$. Therefore D is induced subgraph of $F_n =: D'$.
As a consequence of this theorem we have the corollary.

COROLLARY 1. *For a given $n \geqslant 2$ any digraph D can be embedded as induced subdigraph in a digraph D' having an nth root.*

Proof. Define F as in the proof of Theorem 8 and observe that indeed the following holds: $uv \in D$ if and only if there is a u, v-path in F of length $\leqslant n$ and $u, v \in D$. This means that D is induced subdigraph of $F^n =: D'$.
Now take an undirected graph G. For a given $n \geqslant 2$ construct H as F was constructed from D in Theorem 8. Then G is isomorphic to an induced subgraph of H_n and of H^n. In other words we have the following corollary.

COROLLARY 2. *For a given $n \geqslant 2$ any graph G can be embedded as induced subgraph in a graph G' which is the n-path graph of some graph.*

COROLLARY 3. *For a given $n \geqslant 2$ any graph can be embedded as induced subgraph in a graph G' having an nth root.*

It would have been desirable to find a characterization of graphs (digraphs) having an nth root or of n-path graphs (digraphs) in terms of "forbidden subgraphs" as it has been done, for instance, for linegraphs by Beineke [2]. Theorem 8 and its corollaries show the impossibility of this characterization.

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